

AD-775 445

LINEAR STABILITY OF ECCENTRICALLY
STIFFENED CYLINDRICAL SHELLS UNDER
AXIAL COMPRESSION

Tien-Yu Tsui, et al

Army Materials and Mechanics Research Center
Watertown, Massachusetts

December 1973

DISTRIBUTED BY:

NTIS

National Technical Information Service
U. S. DEPARTMENT OF COMMERCE
5285 Port Royal Road, Springfield Va. 22151

ACCESSION FOR:	
NTIS	<input checked="" type="checkbox"/>
DTIC	<input type="checkbox"/>
1. NAME OF THE REPORT 2. AUTHOR 3. TITLE 4. NUMBER OF PAGES 5. DATE OF PUBLICATION 6. DATE OF ACQUISITION 7. SOURCE 8. PRICE 9. AVAILABILITY STATEMENT 10. DISTRIBUTION STATEMENT 11. SECURITY CLASSIFICATION 12. SECURITY CLASSIFICATION AUTHORITY 13. SECURITY CLASSIFICATION DATE 14. SECURITY CLASSIFICATION EXPIRATION DATE 15. SECURITY CLASSIFICATION REVIEW DATE 16. SECURITY CLASSIFICATION REVIEW AUTHORITY 17. SECURITY CLASSIFICATION REVIEW DATE 18. SECURITY CLASSIFICATION REVIEW AUTHORITY 19. SECURITY CLASSIFICATION REVIEW DATE 20. SECURITY CLASSIFICATION REVIEW AUTHORITY	

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

Mention of any trade names or manufacturers in this report shall not be construed as advertising nor as an official indorsement or approval of such products or companies by the United States Government.

ib

DISPOSITION INSTRUCTIONS

Destroy this report when it is no longer needed.
Do not return it to the originator.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AMMRC TR. 73-58	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) LINEAR STABILITY OF ECCENTRICALLY STIFFENED CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION		5. TYPE OF REPORT & PERIOD COVERED Final Report
7. AUTHOR(s) Tien-Yu Tsui and Chatta Lakshmikantham		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Army Materials and Mechanics Research Center Watertown, Massachusetts 02172 AMXMR-TE		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS D/A Project: 1T161102B33A AMCMS Code: 611102.11.85800 Agency Accession: DA OC4693
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Materiel Command Alexandria, Virginia 22304		12. REPORT DATE December 1973
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 33
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Buckling Cylindrical shells Dynamic structural analysis		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) (SEE REVERSE SIDE)		

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

Block 20.

ABSTRACT

Linear stability of eccentrically stiffened cylindrical shells under axial compression has been investigated in this paper with an aim to establish the parameter ranges for the prevailing buckling modes. Charts delineating the zones of prevailing modes are presented in a parameter plane. Useful approximate relations are given for the bounds on parameters to predict a buckling mode for a shell in the moderate length range.
(Authors)

ia

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

CONTENTS

	Page
NOMENCLATURE	iii
INTRODUCTION	1
GOVERNING EQUATIONS	3
DISCUSSION OF THE BUCKLING EQUATION	9
AXIALLY STIFFENED SHELL	11
Axisymmetric Case	12
Asymmetric Case	13
Outside Stiffeners (e_s negative): Comparison of Solutions . . .	14
Inside Stiffeners (e_s positive): Comparison of Solutions . . .	15
Numerical Results: Discussion of Zones	15
RING STIFFENED SHELL	18
Axisymmetric Solution	18
Asymmetric Case: A Closed Form Solution	18
Comparison of Solutions: Outside Stiffeners (negative e_r) . . .	20
Comparison of Solutions: Inside Stiffeners	20
Numerical Results and Discussion	21
CONCLUSION	26

NOMENCLATURE

A_s, A_r	Cross sectional area of stiffeners
b_s, b_r	Stiffener spacing
E	Young's modulus of shell and stiffener, assumed same
$\bar{E}_s, \bar{E}_r (e_s, e_r)$	Stiffener eccentricity (dimensionless)
f	Dimensionless stress function
G	Shear modulus of skin and stiffener
H	Shell thickness
I_s, I_r	Moment of inertia of stiffener about shell middle surface
J_s, J_r	Torsional constant of stiffener
L	Cylinder length
m, n	Wave numbers in axial and circumferential directions
$M_x, M_y, M_{xy}, M_{yx} (m_x, m_y, m_{xy}, m_{yx})$	Stress couples (dimensionless)
$N_x, N_y, N_{xy} (n_x, n_y, n_{xy})$	Stress resultants (dimensionless)
\hat{n}	Critical load (dimensionless)
R	Radius of shell
$U, V, W (u, v, w)$	Middle surface displacements (dimensionless)
$X, Y (x, y)$	Axial and circumferential coordinates (dimensionless)
Z	Batdorf parameter = L^2/RH
α_s, α_r	Axial stiffness parameter =

$$\left(\frac{1-\nu^2}{h} \right) \left(\frac{A_s}{b_s}, \frac{A_r}{b_r} \right)$$

η_s, η_r

Flexural stiffness parameter =

$$\left(\frac{E}{D} \right) \left(\frac{I_s}{b_s}, \frac{I_r}{b_r} \right)$$

γ_s, γ_r

Torsional stiffness parameter =

$$\left(\frac{G}{D(1-\nu)} \right) \left(\frac{J_s}{b_s}, \frac{J_r}{b_r} \right)$$

ν

Poisson's ratio of shell and stiffeners

s, r

Denote stringers (axial stiffeners)
and rings

comma

Denotes partial differentiation

INTRODUCTION

Research activity in the compressive general instability of stiffened cylindrical shells, in spite of some early work in the 1930's (see for example, References 1 and 2), did not receive sustained effort until the advent of launch and space vehicle designs of the early 60's. In this latter period, extensive test results were obtained on stiffened cylinders which showed that, by and large, these agreed with the predictions of a Donnell-type linear theory which included the stiffeners in the expressions for the resistance of the shell.

Of the various formulations used in this period, we are concerned here with the linear theory of Prof. Singer and his coworkers at Technion,³ which was also proposed by Block et al⁴ in a slightly more generalized form. The main contribution of this theory is that it accounts for the coupling between the bending stress resultants and the middle surface stretching on the one hand and that between the membrane stress resultants and the changes in the middle surface curvature on the other. This coupling, produced by the eccentricity of the stiffener center line location with respect to the shell middle surface, has the net effect of involving odd powers of the eccentricity parameter in the final expression for the buckling stress. Not surprisingly then, the sign of the eccentricity with respect to the middle surface of the shell (i.e., due to the stiffener being outside or inside with respect to the shell) has a profound influence

-
1. DSCHIGU, D. *Die Druckfestigkeit Versteifter Zylindrischer Schalen*. Luftfahrtforschung, v. 11, no. 8, 1935, p. 233-234.
 2. TAYLOR, J. L. *The Stability of Monocoque in Compression*. British R and M, no. 1679, June 1935.
 3. BARUCH, M., and SINGER, J. *Effect of Eccentricity of Stiffeners on the General Instability of Stiffened Cylindrical Shells Under Hydrostatic Pressure*. J. Mech. Eng. Science, v. 5, no. 1, 1963, p. 23-27.
 4. BLOCK, D. L., CARD, M. F., and MIKULAS, M. M. *Buckling of Eccentrically Stiffened Orthotropic Cylinders*. NASA TND-2960, August 1965.

on the buckling load. The change in the buckling load is accentuated the most in the case of axially stiffened cylinders under axial compression. This eccentricity phenomenon is discussed extensively by Singer et al in References 5 and 6.

However, since their main preoccupation has been with this eccentricity effect, certain aspects of the problem, such as the characteristic buckling modes, are not adequately treated in their paper.⁶ For instance, in the case of an axially stiffened cylinder the axisymmetric modes are dismissed with the statement that they are possible for certain combinations of the geometrical parameters. Now, it is a characteristic of the stiffened cylinder buckling that unlike the unstiffened cylinders (where the linear solution is obtained without reference to the buckling modes) the solutions are governed by the characteristic modes: axisymmetric, general asymmetric and for axially stiffened cylinders, an asymmetric mode with a single half-wave along the length of the cylinder.⁷ In fact, for some of the results presented in Reference 6 pertinent to the axial case, the buckling mode is indeed the last mentioned type with a single axial half-wave, though nowhere is it mentioned in the paper.

The ab initio selection of the governing buckling mode for a given stiffener geometry is, perhaps, not very critical in obtaining an answer for the linear problem; however it becomes very important in a related nonlinear problem connected with the dynamic stability of imperfect

-
5. SINGER, J., BARUCH, M., and HARARI, O. *Further Remarks on the Effect of Eccentricity of Stiffeners on the General Instability of Stiffened Cylindrical Shells*. TAE Report No. 42, Technion - Israel Inst. of Tech., Haifa, Israel, Aug. 1965.
 6. SINGER, J., BARUCH, M., and HARARI, O. *On the Stability of Eccentrically Stiffened Cylindrical Shells Under Axial Compression*. Int. J. Solids Structures, v. 3, 1967, p. 445-470.
 7. LAKSHMIKANTHAM, C., GERARD, G., and MILLIGAN, R. *Linear Orthotropic Stability of Shallow-Stiffened Cylinders: An Assessment*. The Aeronautical Quarterly, v. 21, May 1970, p. 133-144.

cylinders. Roth and Klosner⁸ have demonstrated that in the case of dynamic stability of unstiffened imperfect cylinders, the static linear (classical) value forms the limiting solution for zero imperfection. In a similar study with stiffened cylinders the present authors have found^{9,10} that the mode of buckling remains the same in the linear (perfect) and nonlinear (imperfect) ranges, and hence for an efficient determination of the critical dynamic buckling load (that is, from a computer-time point of view), a knowledge of the linear static modes is extremely important.

Having thus established a rationale for reexamining the linear stability problem of an eccentrically stiffened cylindrical shell, we propose to show that from among the numerous geometrical parameters entering the problem, one can rationally select a few which delineate the regions of prevailing buckling modes.

GOVERNING EQUATIONS

Although the buckling equations are similar to those of Singer et al,⁶ our derivation of them is different, being patterned on the nonlinear formulations. Also our final equations are cast into a form which lends itself to an easier examination of the relative importance of the various stiffening parameters.

8. ROTH, R. S., and KLOSNER, F. *Nonlinear Response of Cylindrical Shells Subjected to Dynamic Axial Loads*. A.I.A.A. Journal, v. 2, Oct. 1964, p. 1788-1794.

9. TSUI, TIEN-YU, and LAKSHMIKANTHAM, C. *The Nonlinear Dynamic Response of Stiffened Shells Under Compressive Axial Impulse*. Presented at the 4th Canadian Congress of Applied Mechanics, Montreal, Canada, 1973.

10. LAKSHMIKANTHAM, C., and TSUI, TIEN-YU. *Dynamic Stability of Axially Stiffened Imperfect Cylindrical Shells Under Axial Impulsive Loading*. Accepted for Publication by AIAA Journal.

The assumptions basic to the linear problem are similar to those of Reference 6: a) the stiffeners are close enough that they can be "smeared" over the shell; b) the stiffeners are essentially beam-like elements contributing to the membrane and bending resistance of the shell but not to the shear resistance; c) the entire shell including the stiffeners is activated in the buckling; that is, local instability failures are ruled out.

Figure 1 describes the basic geometric parameters of the shell. With the normal to the shell surface taken as positive inward, positive (negative) values of \bar{E}_s , \bar{E}_r indicate inside (outside) location of the stiffeners with respect to the shell.

The unstiffened shell is characterized by three parameters L , R , h (length, radius, and the thickness of the shell) which can be combined into the so-called Batdorf parameter $Z = L^2/Rh$, the ranges of whose values are used to describe a "short" or a "moderate length" cylinder.

The effective increases in the area of section, the section area

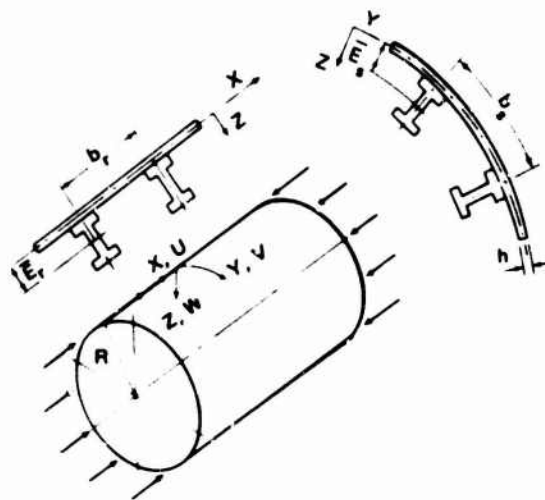


Figure 1. Shell Geometry

moment, and the section torsional stiffness of the shell due to the presence of stiffeners are best characterized by nondimensional parameters α , η and γ with subscripts s or r to denote a stringer (axial stiffener) or a ring (frame or circumferential stiffener). These parameters are the ratios of the properties of the stiffener section relative to a unit shell section between the stiffener center lines. We have assumed that the stiffeners have the same material properties as those of the shell; otherwise the relative stiffener-to-shell moduli ratios are to be included in the definition of α , η and γ .

Although these parameters are not independent of each other, it is still convenient to consider them so, from a computational point of view. For slender and deep stiffeners of rectangular sections (such as is assumed here) one can express γ in terms of α multiplied by a constant. Hence the chief stiffener parameters of interest are α , η together with \bar{E} the eccentricity of the stiffener.

In order to derive the basic equations, we introduce the following nondimensional quantities:

All lengths (including the coordinate distances X , Y) are normalized with respect to R , the shell radius. Thus we write:

$$[u, v, w, x, y, \bar{e}_s, \bar{e}_r] = [U, V, W, X, Y, \bar{E}_s, \bar{E}_r]/R \quad (1)$$

Dimensionless stress and moment resultants are given by:

$$n_x, n_y, n_{xy}, \bar{n}_x = [N_x, N_y, N_{xy}, \bar{N}_x]/B \quad B = Eh/(1-\nu^2) \quad (2)$$

$$m_x, m_y, m_{xy}, m_{yx} = [M_x, M_y, M_{xy}, M_{yx}]/D \quad D = Eh^3/12(1-\nu^2)$$

The linear strain-displacement relationships at the middle surface of shell are:

$$\epsilon_x^o = u_{,x} ; \epsilon_y^o = v_{,y} - w ; \epsilon_{xy}^o = 1/2 (u_{,y} + v_{,x}) \quad (3)$$

$$R\kappa_x = w_{,xx} ; R\kappa_y = w_{,yy} ; R\kappa_{xy} = w_{,xy} = R\kappa_{yx} \quad (4)$$

The stress-resultant strain relationships are given by:

$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \\ m_{yx} \end{bmatrix} = \begin{bmatrix} (1+\alpha_s) & \nu & 0 & -\bar{e}_s R \alpha_s & 0 & 0 & 0 \\ \nu & (1+\alpha_r) & 0 & 0 & -\bar{e}_r R \alpha_r & 0 & 0 \\ 0 & 0 & (1-\nu) & 0 & 0 & 0 & 0 \\ \hline -\bar{e}_s \beta_s & 0 & 0 & (1+\eta_s) & \nu & 0 & 0 \\ 0 & -\bar{e}_r \beta_r & 0 & \nu & (1+\eta_r) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-\nu)(1+\gamma_s) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (1-\nu)(1+\gamma_r) \end{bmatrix} \begin{bmatrix} \epsilon_x^o \\ \epsilon_y^o \\ \epsilon_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \\ \kappa_{yx} \end{bmatrix} \quad (5)$$

M_1

M_2

M_3

M_4

The nonzero submatrices M_2 and M_3 in Eq. (5) provide the coupling effects of the stiffener eccentricity.

In Eq. 5 β_s (β_r) is not a new parameter but is related to α_s (α_r) through $D\beta_s = B\alpha_s$.

The governing equations are then given by:

$$n_{x,x} + n_{xy,y} = n_{xy,x} + n_{y,y} = 0 \quad (6a, 6b)$$

$$- \frac{1}{12} \frac{H^2}{R^2} L_3^2 [w] + \alpha_s \bar{e}_s \epsilon_{x,xx}^o + \alpha_r \bar{e}_r \epsilon_{y,yy}^o - \bar{n}_x w_{,xx} + n_y = 0 \quad (6c)$$

where L_3 is a linear differential operator given by

$$L_3 () = (1 + \eta_s) (),_{xxxx} + 2 \left\{ 1 + \frac{(1-\nu)}{2} (\gamma_s + \gamma_r) \right\} (),_{xxyy} + (1 + \eta_r) (),_{yyyy} \quad (6d)$$

and \bar{n}_x is the (dimensionless) external compressive load.

Eq. (6c) reduces to the standard (linear) form when all stiffness and eccentricities are neglected.

Eqs. (6a, 6b) are identically satisfied by introducing a stress function of such that $n_x = f_{,yy}$; $n_y = f_{,xx}$ and $n_{xy} = -f_{,xy}$.

Making use of the stress-strain relationships Eq. (5) and introducing the stress function f , Eq. (6c) may be rewritten as:

$$-\frac{1}{12} \frac{H^2}{R^2} L_3 (w) - \frac{1}{A_{rs}} L_2 (f) + \frac{1}{A_{rs}} L_4 (w) - \bar{n}_x w_{,xx} + f_{,xx} = 0 \quad (7)$$

where L_2 and L_4 are linear operators given by

$$L_2 () = \nu \alpha_s \bar{e}_s (),_{xxxx} - \{ (1 + \alpha_s) \alpha_r \bar{e}_r + (1 + \alpha_r) \alpha_s \bar{e}_s \} (),_{xxyy} + \nu \alpha_r \bar{e}_r (),_{yyyy} \quad (8a)$$

$$\text{and } L_4 () = (1 + \alpha_r) (\alpha_s \bar{e}_s)^2 (),_{xxxx} - 2\nu \alpha_s \alpha_r \bar{e}_s \bar{e}_r (),_{xxyy} + (1 + \alpha_s) (\alpha_r \bar{e}_r)^2 (),_{yyyy} \quad (8b)$$

$$\text{and } A_{rs} = (1 + \alpha_s) (1 + \alpha_r) - \nu^2$$

A second independent equation involving f and w is provided by the compatibility relation which is easily deduced from Eq. (3) to be

$$\epsilon_{x,yy}^o + \epsilon_{y,xx}^o - 2\epsilon_{xy,xy}^o = -w_{,xx} \quad (9)$$

Eq. (9) can be written in terms of f and w by again making use of Eq. (5).

This results in

$$L_1(f) = L_2(w) - A_{rs} w_{,xx} \quad (10)$$

where $L_1(\quad) = (1 + \alpha_s)(\quad)_{,xxxx} + 2\left(\frac{A_{rs}}{1-\nu}\right)(\quad)_{,xyxy} + (1 + \alpha_r)$

$$(\quad)_{,yyyy} \quad (11)$$

Thus the basic equations of the static buckling problem are Eqs. (7) and (10).

Now we notice that the operators L_1, L_2, L_3, L_4 are linear with constant coefficients (shifted biharmonic types).

Since the buckling modes are essentially harmonic, w and consequently f are both harmonic. Hence we see that the following operations with the operators hold:

$$L_i[L_j(\phi)] = L_j[L_i(\phi)] \quad (12)$$

$$\text{and also } L_i[(\quad)_{,xx}] = [L_i(\quad)]_{,xx} \quad i, j = 1, 2, 3, 4 \quad (13)$$

Thus we can eliminate f from Eq. (7) by operating with L_1 upon Eq. (7) and as a result have

$$-\frac{1}{12} \frac{H^2}{R^2} L_1 L_3(w) - \frac{1}{A_{rs}} L_1 L_2(f) + \frac{1}{A_{rs}} L_1 L_4(w) - \bar{n}_x L_1(w_{,xx}) + L_1 f_{,xx} = 0 \quad (14)$$

By making use of Eqs. (12), (13) and (10) we finally have

$$\frac{1}{12} \frac{H^2}{R^2} L_1 L_3(w) - \frac{1}{A_{rs}} [L_1 L_4(w) - L_2 L_2(w)] + A_{rs} w_{,xxxx} - 2 L_2(w_{,xx}) + \bar{n}_x L_1(w_{,xx}) = 0 \quad (15)$$

Eq. (15) is thus the Donnell equation for the eccentrically stiffened shell problem with only w appearing in it.

We seek a solution of the above by choosing the following form for

w:

$$w = \bar{w} \cos m\pi x \cos ny \quad (16)$$

where

$$\bar{\pi} = nR/L$$

Using Eq. (16) in Eq. (15) we find

$$\frac{\bar{n}}{A_{rs}} \frac{R}{H} \hat{n} = \frac{(m\pi)^2}{A_{rs}} \frac{1}{Z} \left[\frac{C_{rs}}{12} - \frac{1}{A_{rs}} \left\{ D_{rs} - \frac{B_{rs}^2}{P_{rs}} \right\} \right] + \frac{Z}{(m\pi)^2} \frac{1}{P_{rs}} + \frac{2B_{rs}}{A_{rs} P_{rs}} \quad (17)$$

$$\text{where } C_{rs} = (1 + \eta_s) + 2 \left\{ 1 + \frac{(1-\nu)}{2} (\gamma_s + \gamma_r) \right\} \beta^2 + (1 + \eta_r) \beta^4$$

$$D_{rs} = (\alpha_s e_s)^2 (1 + \alpha_r) - 2\nu \alpha_s \alpha_r e_s e_r \beta^2 + (1 + \alpha_s) (\alpha_r e_r)^2 \beta^4$$

$$B_{rs} = \nu \alpha_s e_s - \{ (1 + \alpha_s) \alpha_r e_r + (1 + \alpha_r) \alpha_s e_s \} \beta^2 + \alpha_r e_r \nu \beta^4$$

$$P_{rs} = (1 + \alpha_s) + 2 \left(\frac{A_{rs}}{1-\nu} - \nu \right) \beta^2 + (1 + \alpha_r) \beta^4$$

$$\beta = \frac{n}{m\pi}; e_s, e_r = \left(\frac{\bar{E}_s}{H}, \frac{\bar{E}_r}{H} \right) = \frac{R}{H} (\bar{e}_s, \bar{e}_r) \quad (18)$$

The buckling load is obtained by minimizing Eq. (17) with respect to the wave numbers n, m.

In deriving Eq. (17), we notice that no emphasis has been placed on the boundary conditions. This is as it should be. It is inherent in the classical shell stability theory that the eigenvalues are obtained without reference to boundary conditions.

DISCUSSION OF THE BUCKLING EQUATION

Eq. (17) constitutes the fundamental equation from which the (minimum) buckling load is obtained for a given shell with prescribed geometry and, furthermore, it can be demonstrated that it is identical with the equation

used by Singer et al.⁶ We now proceed to deduce several interesting results based on the form of Eq. (17) together with the stiffening parameters defined in Eq. (18).

Firstly, we notice that the eccentricity parameters e_s , e_r appear explicitly only in the terms B_{rs} and D_{rs} . Thus B_{rs} and D_{rs} are the coupling effects due to eccentricity. If the eccentricities were ignored, (either owing to symmetrical stiffeners or as a higher order effect) B_{rs} and D_{rs} would drop out, and the resulting expression

$$\hat{n} = \frac{1}{12} \frac{(m\pi)^2}{Z} \frac{C_{rs}}{A_{rs}} + \frac{Z}{(m\pi)^2} \frac{1}{P_{rs}} \quad (19)$$

would admit positive (and hence physically admissible) solutions for all choices of α and η .

However, B_{rs} involves first powers of e_s , e_r while D_{rs} is quadratic in them. Hence the sign of the last term in Eq. (17) is dictated by the sign of the eccentricity terms. This brings out a distinctive feature of the problem of an eccentrically stiffened shell. The presence of a minus sign within the square brackets, together with the possibility that the last term could be negative, makes only certain combinations of the stiffening parameters yield physically admissible solutions. It is not hard to see, given the structure of Eq. (17), that there are admissible combinations of the parameters yielding negative or zero values for the buckling load! It is even more remarkable that the same set of physical parameters which would yield admissible (positive) solutions based on Eq. (19) would cease to do so if the symmetry of the stiffeners is distorted.

A second observation which we can make, and to which we shall come back, is the absence of Z explicitly in the last term of Eq. (17), which makes certain solutions valid for all Z ranges.

Eq. (17), involving both rings and stringers, is applicable to a general (grid) stiffened shell. Particular cases of axial or ring stiffeners are obtained from Eq. (17) by setting the appropriate stiffening terms equal to zero. A grid stiffened shell is essentially a composite of axial and ring stiffened cylinders, and hence its behavior is understood best by studying the axial and the ring cases separately. This paper confines its attention to the axial and ring cases only.

AXIALLY STIFFENED SHELL

All the parameters with subscript r are set equal to zero in Eqs. (17) and (18) and after some manipulation we can obtain the governing equation for the axially stiffened shell under axial compression as follows:

$$\frac{1}{n} = \frac{1}{A_s} \left[\frac{(m\pi)^2 C_s}{Z} - \frac{(\alpha_s e_s)^2 (1 + \frac{2\beta^2}{1-\nu})}{P_s} \right] + \frac{Z}{(m\pi)^2} \frac{1}{P_s} + \frac{2\alpha_s e_s (\nu - \beta^2)}{A_s P_s} \quad (20)$$

where

$$\begin{aligned} C_s &= (1 + \eta_s) + 2 \{1 + (1-\nu) \gamma_s / 2\} \beta^2 + \beta^4 \\ P_s &= (1 + \alpha_s) + 2 \{1 + \alpha_s / (1-\nu)\} \beta^2 + \beta^4 \\ A_s &= (1 + \alpha_s) - \nu^2 \end{aligned} \quad (21)$$

For a shell of given α , e , η and Z , Eq. (20) yields a load for each integer value of m, n such that $m \geq 1$; $n \geq 0$. ($m=n=0$ case corresponds to

a constant w displacement which is of no interest to the buckling problem). The lowest of these loads for all m, n combinations is the buckling load for the given shell.

From a theoretical point of view, however, it is more convenient to assume ab initio a buckling mode and investigate the range of its validity. We first consider the axisymmetric case ($n=0$).

Axisymmetric Case

This is obtained by setting $\beta=0$ in Eqs. (20) and (21). The resulting equation is then minimized with respect to m to yield the following:

$$\hat{n}_{axi} = 2 \left\{ \frac{1}{A_s} \frac{1}{(1 + \alpha_s)} \left[\frac{1 + \eta_s}{12} - \frac{(\alpha_s e_s)^2}{1 + \alpha_s} \right] \right\}^{1/2} + \frac{2\alpha_s e_s}{A_s (1 + \alpha_s)} \quad (22)$$

The sign of the last term in Eq. (22) depends upon the sign of e_s . We can thus distinguish the two cases of inside (+ve e_s) and outside (-ve e_s) stiffeners.

We notice that for real values of \hat{n}_{axi} ,

$$\eta_s \geq 12 (e_s \alpha_s)^2 / (1 + \alpha_s) - 1 \equiv \eta_+^* \quad (23)$$

should hold in Eq. (22). It is interesting to note that for η_s greater than η_+^* such that the square root term is positive, the predicted load is affected by the sign of the last term. Thus, for e_s positive the load is higher than that with e_s negative. Hence the inside stiffener case would yield higher axisymmetric loads than outside stiffeners.

It follows that for negative e_s , the minimum load is zero and the corresponding η_s value is given by:

$$\eta_s = \eta_-^* = 12 (e_s \alpha_s)^2 / A_s - 1 \text{ (with } \hat{n}_{axi} (-) = 0) \quad (24)$$

η_-^* is greater than η_+^* as $A_s (\equiv 1 + \alpha_s - v^2) < 1 + \alpha_s$ for positive

v . However the difference between the two η values is very little.

For very large α values the difference is of the order e_s^2 .

An important feature of the axisymmetric solution is that it is independent of Z . Hence if the solution is valid it will hold for both 'short' and 'moderate-length' cylinders.

Asymmetric Case

The asymmetric solution is obtained from Eq. (20) as the lowest possible load for all possible integer combinations of m and n ($n > 0$) for all given values of α , e , η , and Z . Obviously, a large number of computations has to be performed to cover a wide range of these parameters. However, a slightly modified form of Eq. (20) is more convenient for theoretical comparison with the axisymmetric solution Eq. (22). This approximate form is obtained by noting that for the axisymmetric case of a moderate length cylinder, generally $n \gg m$ and hence β^2 can be taken much larger than unity in Eq. (20) and thus we write:

$$\hat{n} \approx \frac{1}{A_s} \frac{(m\pi)^2}{Z} \left[\frac{C_s}{12} - \frac{2 (\alpha_s e_s)^2 \beta^2}{P_s} \right] + \frac{Z}{(m\pi)^2} \frac{1}{P_s} - \frac{2 \alpha_s e_s}{A_s P_s} \quad (25)$$

Even further simplification is possible if we compare the sizes of some of the terms with β^4 in Eqs. (20) and (21). Generally, α and e have numerically the smallest range of values, seldom exceeding a number such as 10 for practical cylinders; however, η has a wide range and can have values of the order 10^4 . Thus retaining terms of the order β^4 in Eqs. (20) and (21) we can rewrite Eq. (25) as:

$$\hat{n} = \frac{1}{A_s} \frac{(m\pi)^2}{Z} \left[\frac{1 + \eta_s + \beta^4}{12} \right] + \frac{Z}{(m\pi)^2} \frac{1}{\beta^4} - \frac{2 \alpha_s e_s}{A_s \beta^2} \quad (26)$$

Certain features of the axisymmetric solution in comparison with the axisymmetric solution Eq. (22) became quite apparent from the form of Eq. (26). Firstly, we notice that the sign of the last term in Eq. (26) is opposite to that in Eq. (22). Thus the stiffener effects is reversed in the asymmetric case: buckling load for e_s positive (inside stiffeners) is lower than that for negative e_s . Secondly, the solution of Eq. (26) is strongly Z -dependent. Also in Eq. (26) there is no apparent lower limit to n values in contrast to the n^* values of Eqs. (23) and (24); hence in the regions where the axisymmetric solution is inadmissible, the asymmetric solution will prevail. Consequently, some valid comparisons can be readily made between the two modes; however, it is convenient to distinguish the two cases of eccentricity.

Outside Stiffeners (e_s negative): Comparison of Solutions

For negative e_s , all the terms of Eq. (26) are positive, whereas the minimum \hat{n}_{axi} is zero from Eq. (24). Hence we suspect that for any α, e combination, in the vicinity of n_-^* (i.e. $\eta_s \geq \eta_-^*$), given from Eq. (24), the axisymmetric solution will prevail. For values of η_s much larger than η_-^* , Eq. (22) shows a monotone behavior and for very large η_s values, Eq. (22) will approach the limit.

$$\bar{n}_{axi} = Z \left(\frac{1 + \eta_s^{1/2}}{1 + \alpha_s} \right) \left(\frac{1}{12 A_s} \right)^{1/2} \quad (27)$$

However for large η_s values, Eq. (26) is still modified by Z and other factors. Intuitively, we expect, then, that the asymmetric solutions will prevail for large η_s values. Again for η_s lower than the critical value for axisymmetric solution ($=\eta_s^*$), the asymmetric solution being admissible, we can assume that we will have the asymmetric solution. Thus, in an $\eta_s - \alpha_s$ plane (for fixed values of e_s), we expect two zones of asymmetric modes interspersed by a (narrow) zone where axisymmetric solution prevails.

Inside Stiffeners (e_s positive): Comparison of Solutions

Here, from Eq. (26) we see that \hat{n} values can decrease to zero whereas the lowest possible axisymmetric load (occurring at $\eta_s = \eta_s^*$) is seen from Eq. (22) to be

$$\hat{n}_{-axi} (+) = \frac{Z \alpha_s e_s}{A_s (1 + \alpha_s)} \quad (28)$$

Hence we can expect that there may be no axisymmetric zone at all for the Z values in the moderate length region.

Numerical Results: Discussion of Zones

In order to give substance to the above arguments, a number of α , e , η combinations were used to study Eqs. (20) and over a wide range of Z values from 10^2 to 10^4 α was varied from 0 to 10; e was taken as $\pm 1, 3, 5, 7$, and 10. For each combination of the parameters, a minimum of Eq. (20) was found by considering integer values of m (1 to 20) and n (1 to 50) on a Univac 1106 computer.

The results are shown in Figure 2. For every case the η - α diagram shows four typical zones: Zone I of asymmetric $m=1$ mode; Zone II of Z-dependent axisymmetric and asymmetric ($m=1$) modes; Zone III of axisymmetric mode and Zone IV of asymmetric mode where generally $m \neq 1$.

Zone I occurs for Z generally in the moderate length range, though this range decreases with increasing e . Thus, for the highest e considered, i.e. $e = 10$, the Z range of Zone I was from 150 to 10^4 while for $e=1$, Z could be as low as 20.

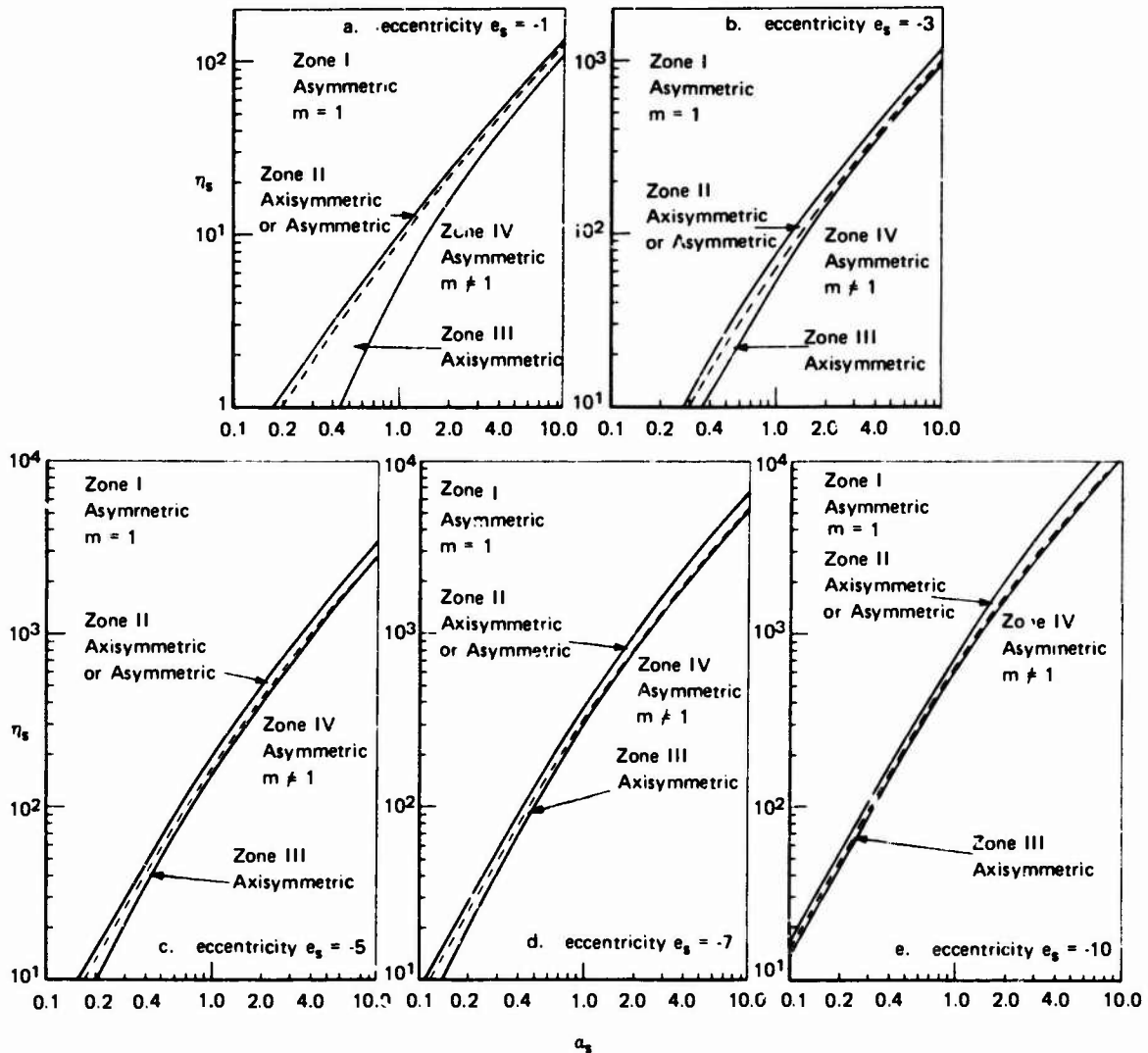


Figure 2. Governing Buckling Modes in Various Zones for Axially Stiffened Cylinder Under Axial Compression

Zone II is a transition zone where axisymmetric solutions govern for Z values which increase with decreasing η , until Zone III values of η are reached, where axisymmetric solutions governs all Z values. Generally Zones II and III form a narrow band and the curve distinguishing the two becomes asymptotic to the curves of Zone I and Zone III. The lower bound of Zone III is the η^* curve defined in Eq. (24).

For positive e values also, there exist the same four zones though the middle zones are even less prominent than in the negative e case. It is interesting to note that the same curve forms the lower bound of Zone I in both the positive and negative cases. It may also be mentioned that the values of shell parameter studied by Singer et al⁶ fall within Zone I for the axially stiffened case.

Summarizing the axially stiffened case, we have for every α, e combination a minimum η above which the buckling is in asymmetric $m=1$ mode, valid for the shell parameter Z with a minimum range of 150 to 10^4 ; for lower η values, axisymmetric solutions prevail up to a critical value of η^* below which once again asymmetric modes (though $m \neq 1$ now) occur.

From a predictive point of view, where we wish to design a cylinder to buckle in the $m=1$ mode, we are obviously interested in the bounding curve of Zone I. The following approximate expression was found to provide a close upper bound on the curve:

$$\eta = 18 \frac{(\alpha_s e_s)^2}{1 + \alpha_s} \quad (29)$$

valid for all the ranges of the parameters tested.

RING STIFFENED SHELL

For the ring stiffened shell, we set all terms with subscript s equal to zero in Eqs. (17) and (18) and after some manipulation obtain

$$\hat{n} = \frac{(m\pi)^2}{A_r Z} \left[\frac{C_r}{1-\nu} - (\alpha_r e_r)^2 \beta^4 \left\{ \frac{\beta^4 + 2\beta^2}{1-\nu} \right\} \right] + \frac{Z}{(m\pi)^2} \frac{1}{P_r} - \frac{2\alpha_r e_r \beta^2 (1-\nu\beta^2)}{A_r P_r} \quad (30)$$

where

$$\begin{aligned} C_r &= 1 + 2 \left(1 + \frac{1-\nu}{2} \gamma_r \right) \beta^2 + (1 + \eta_r) \beta^4 \\ P_r &= 1 + 2 (1 + \alpha_r / 1-\nu) \beta^2 + (1 + \alpha_r) \beta^4 \\ A_r &= 1 + \alpha_r - \nu^2 \quad \beta = n / m\pi \end{aligned} \quad (31)$$

Axisymmetric Solution

The axisymmetric solution is readily obtained by setting $\beta=0$ in Eqs. (30) and (31) and minimizing the resulting expression with respect to m . Thus we have:

$$\hat{n}_{\text{axi}} = 2 / (12 A_r)^{1/2} = 1 / [3(1 + \alpha_r - \nu^2)]^{1/2} \quad (32)$$

Notice that this solution is independent of both η and e .

Asymmetric Case: A Closed Form Solution

The asymmetric case is based on Eq. (30) by the usual minimization with respect to m, n ($n > 0$). However, it is possible to obtain a simple closed form solution for the particular case of $m=1$ and $\beta^2 \gg 1$. Thus by retaining terms of the order β^4 in Eqs. (30) and (31), we get the following approximate expression:

$$\hat{n} \approx \frac{(\pi)^2}{A_r Z} \beta^4 \left[\frac{1 + \eta_r}{12} - \frac{(\alpha_r e_r)^2}{1 + \alpha_r} \right] + \frac{Z}{\pi^2} \frac{1}{(1 + \alpha_r) \beta^4} + \frac{2\alpha_r e_r}{A_r (1 + \alpha_r)} \quad (33)$$

which yields the following minimum

$$\hat{n} = 2 \left[\frac{1}{A_r} \frac{1}{(1 + \alpha_r)} \left\{ \frac{1 + \eta_r}{12} - \frac{(\alpha_r e_r)^2}{1 + \alpha_r} \right\} \right]^{1/2} + \frac{2\alpha_r e_r}{A_r (1 + \alpha_r)} \quad (34)$$

It is instructive to examine Eq. (34). We notice that the form of Eq. (34) is identical with the axisymmetric solution for the axially stiffened case, Eq. (22). This solution again leads to possible zero value for negative e_r (outside stiffeners). Hence we can suspect that for η_r values very close to (and greater than)

$$\eta_r^* (-) = \frac{12 (\alpha_r e_r)^2}{A_r} - 1 \quad (35)$$

the asymmetric $m=1$ mode will prevail for at least the negative e_r case.

Furthermore, for $\eta_r < \eta_r^* (-)$ we see that Eq. (34) will not hold as it will yield negative values for \hat{n} ; hence for $\eta_r < \eta_r^*$ we should expect other modes, more general asymmetric or the axisymmetric modes, to prevail.

Before we try to compare the solutions, we can examine the consequences of letting m be of the same order as n in Eq. (30); the essential result is that $\beta^2 < 1$ and the last term in Eq. (30) gets modified to

$$\hat{n} = \frac{(m\pi)^2}{A_r Z} \left[\frac{C_r}{12} - (\alpha_r e_r)^2 \beta^4 \left\{ \frac{\beta^4 + 2\beta^2}{1 - \nu} \right\} \right] + \frac{Z}{(m\pi)^2} \frac{1}{P_r} - \frac{2\alpha_r e_r \beta^2}{A_r P_r} \quad (36)$$

which is additive for negative e_r and subtractive for positive e_r .

Comparison of Solutions: Outside Stiffeners (negative e_r)

Of the three possible solutions, we can anticipate that for values of η slightly in excess of η_r^* given by Eq. (35), Eq. (34) will provide the lowest possible values; when η is substantially larger than η_r^* , the monotonic behavior of Eq. (34) suggests that the only possible solutions will be the axisymmetric case of Eq. (32) or the more general asymmetric case corresponding to Eq. (36). However, here again since the last term is additive for negative e_r , and also that C_r will be quite dominant, one can intuitively opt for the axisymmetric case which is independent of η and Z . For values of η below η_r^* , the only solutions in contest are the axisymmetric or the general asymmetric. Here again, since C_r may itself be small, the terms in the square bracket may be substantially low to warrant the prevalence of the general asymmetric solution over the axisymmetric solution.

Comparison of Solutions: Inside Stiffeners

For inside stiffeners, with e_r positive, Eq. (36) could give us substantially lower values as the last term is negative, compared to the $m=1$ solution, Eq. (34), where the last term is positive. Hence the probable modes are the axisymmetric and the general asymmetric modes. A further careful examination of Eq. (36) shows that it is strongly dependant on Z , generally increasing with Z (as $\beta^2 < 1$ and all the terms, such as C_r , P_r tend to a limit). Hence for high Z values Eq. (36) may yield high values of \hat{n} which are higher than the axisymmetric. For lower values of Z , $Z/(m\pi)^2$ will be substantially smaller and the sizes of the positive and negative terms will be of the same order; thus yielding quite low values.

Numerical Results and Discussion

Eq. (30) was studied for several combinations of α , e , and η over a range of Z from 10^2 to 10^4 and m, n values in the range (1 to 50) each. Eq. (32), being a function of α only, was computed for all the α values of interest.

The role on stiffener eccentricity on Eq. (30) was found to be that, for negative e (outside stiffeners), \hat{n} (i.e., the asymmetric load) was relatively independent of Z , depending only on η for a shell of given α , e .

However, for positive e , \hat{n} was strongly dependent on both Z and η . Since this difference in behavior distorts the regions of asymmetric and axisymmetric behavior it is convenient to treat the two cases of eccentricity separately.

a) Outside Stiffeners: The solutions of Eq. (30) for negative e were almost independent of Z (in the range of interest) and increased rapidly with η so that for every α , e case it was easy to find a minimum η above which the asymmetric solution of Eq. (30) was always higher than the axisymmetric solution.

Figure (3) shows a typical η - α diagram for a suitable $|e|$ value. The diagram consists of two major zones of axisymmetric and general asymmetric modes. The dividing curve happens to coincide with the η_r^* (-) curve given in Eq. (35), which is based on an $m=1$ solution. This mode is essentially confined to the η^* curve; for values of η even slightly larger than η^* we get only axisymmetric solution as the buckling mode.

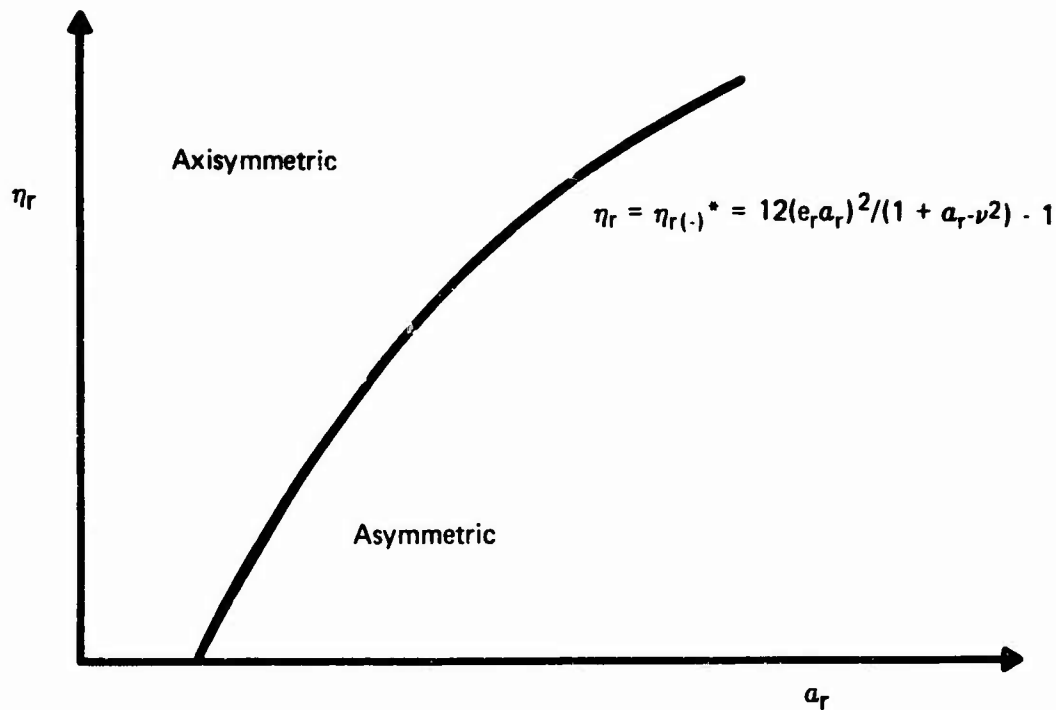


Figure 3. Governing Buckling Modes for Cylinder with Outside Radial Stiffeners Under Axial Compression

Hence from a predictive point of view any value of η slightly larger than η_r^* (-) would give us the minimum combinations for an axisymmetric buckling.

b) Inside Stiffeners: For positive e , the dependence of Eq. (30) on Z exhibits itself in a remarkable way. For $Z \leq 10^3$ generally \hat{n} was independent of Z , but in the range $10^3 > Z > 10^4$ it almost increases monotonically with Z . This behavior is generally true for low η as well as high η as shown in an example in Figure 4. (for a combination of α , e which could be taken as light stiffening). The axisymmetric line (being independent of Z , η , e) is also shown for the α considered. Thus it becomes clear that for $Z > 10^3$ we can have axisymmetric buckling (provided we have a minimum η required). For every case studied we

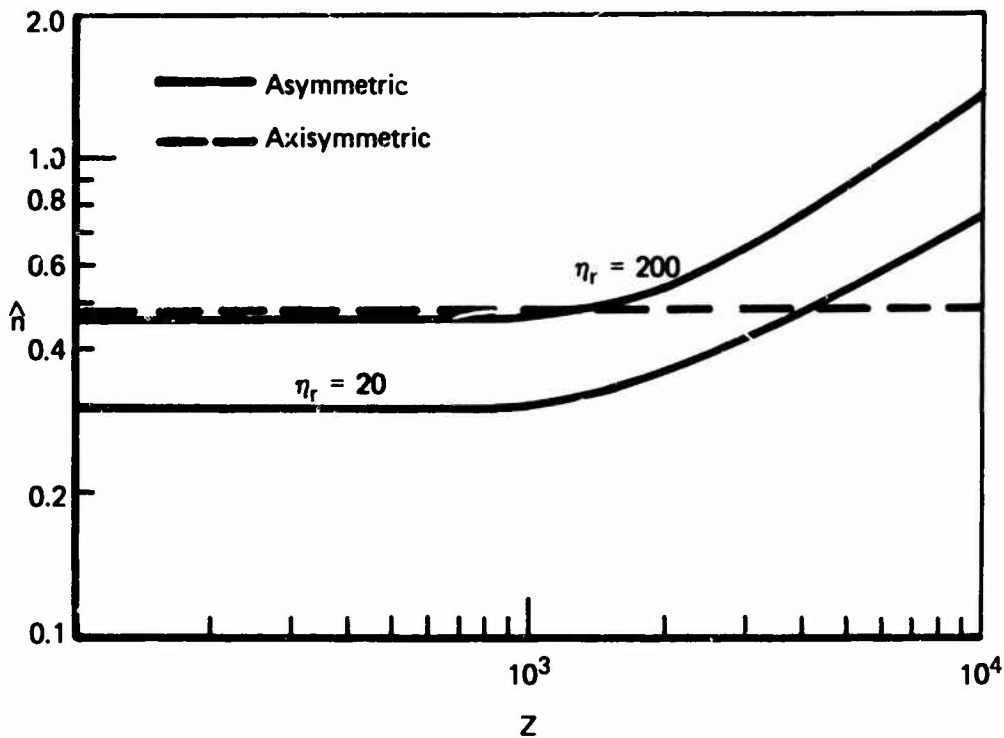


Figure 4. Variation of Buckling Loads Versus Z for Cylinder with Inside Radial Stiffeners ($\alpha_r = 0.5$, $e_r = 3$)

found that for $Z > 10^3$, we could find acceptable η values beyond which axisymmetric solutions become possible. It is also clear from Figure 4 that the higher the Z value the minimum η for axisymmetric buckling becomes lower. Figure 5 shows the minimum η values for a shell of $Z = 2000$.

The dividing curve between the axisymmetric and the asymmetric solution in Figure 5 can be represented by

$$\eta = C \frac{12 (\alpha_r e_r)^2}{1 + \alpha_r} \quad (37)$$

where the values of C are given in Figure 6 for different values of e .

Returning to Figure 4, we notice that for $Z \leq 10^3$ the axisymmetric

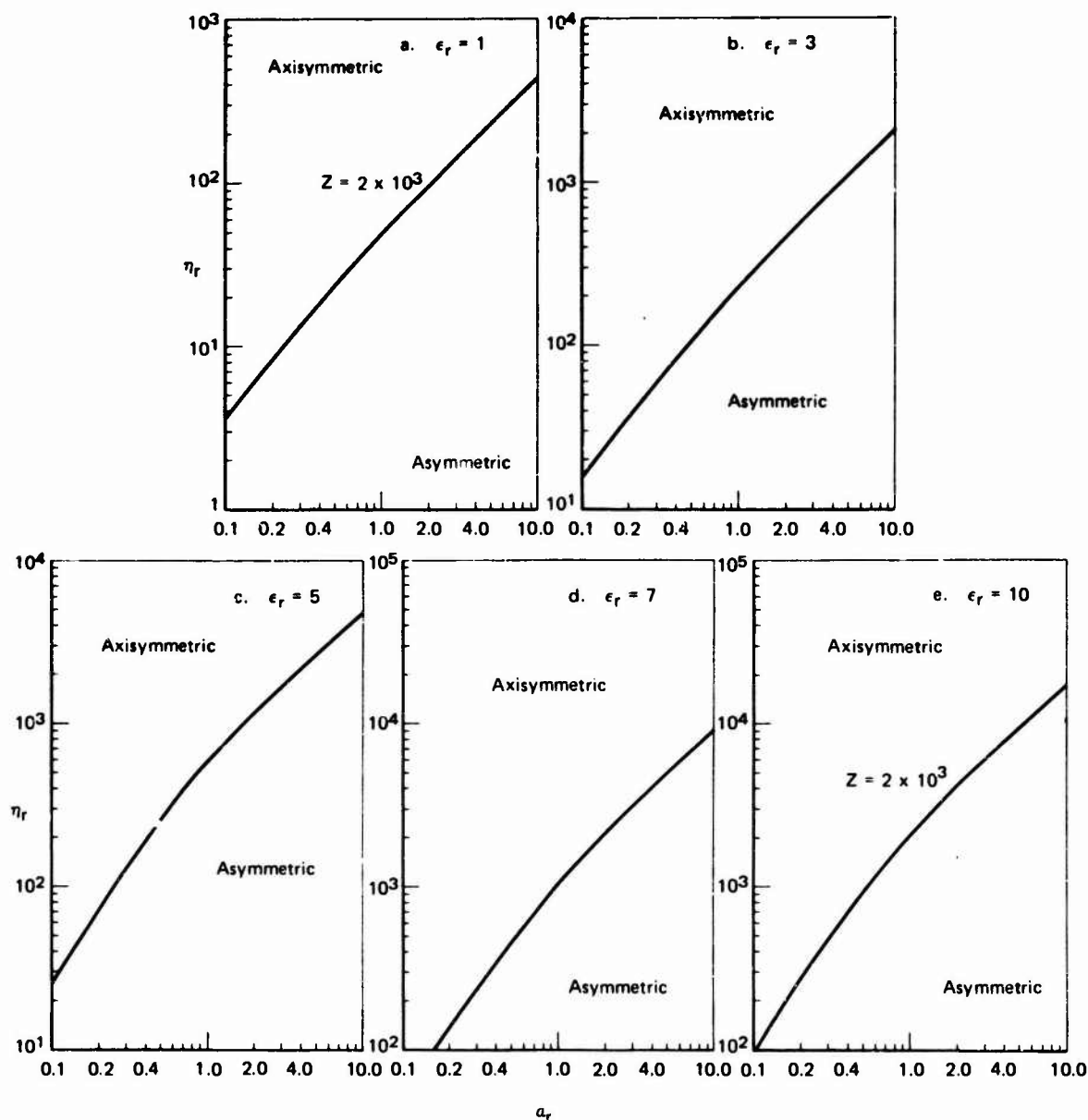


Figure 5. Governing Buckling Modes for Radially Stiffened Cylinder Under Axial Compression

solution was independent of Z , but that for the particular η shown (200) it was slightly below the axisymmetric value. However, no matter how high the value of η was chosen, the asymmetric solution for $Z \leq 10^3$ did not exceed the axisymmetric line. Thus a second remarkable feature of the positive eccentricity emerges: for $Z \leq 10^3$ the axisymmetric solution seems to be almost impossible. The reason is probably that

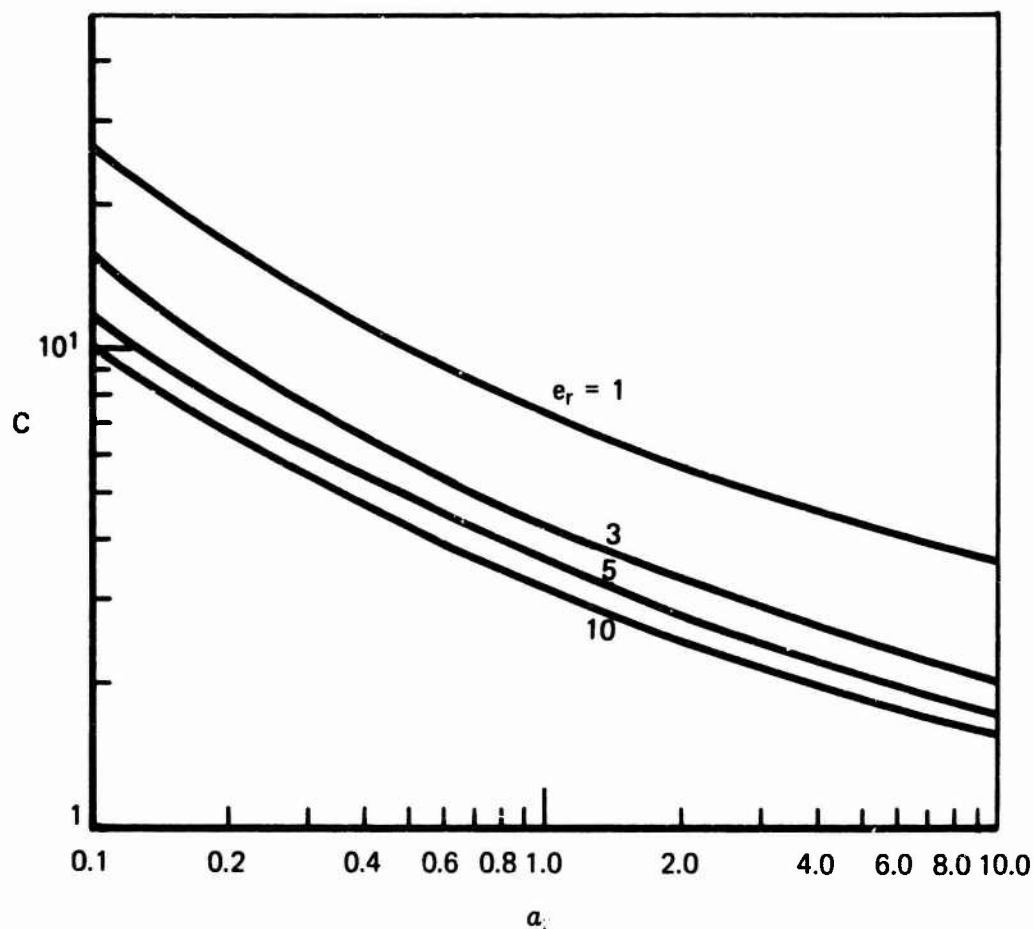


Figure 6. Graphs for Constant C in Equation (37)

for $Z \leq 10^3$ the buckling is a 'many-bayed' phenomenon with each bay buckling as a short cylinder.⁶ However, a point remains: the axisymmetric solution is certainly possible for $Z > 10^3$ which Reference 6 seems to regard as impossible for any Z .

A further remark is also relevant: the axisymmetric solution, being so close to the asymmetric solution (at least for the η shown in Figure 4), provides a close approximation for the buckling load for $10^2 < Z < 10^3$ for a great number of α , e , η combinations.

Summarizing the ring-stiffened case, both axisymmetric and asymmetric buckling are possible and for outside stiffeners it occurs for all Z ranges while for inside stiffeners it is confined to moderate-to-long Z ranges.

CONCLUSION

The primary aim of this paper has been to concentrate on the effect of stiffener eccentricity on the buckling modes rather than on the buckling loads; to present as simply as possible the parameter combinations for which a particular mode would prevail. In some cases we have succeeded in presenting a simplified relationship between the parameters which would give minimum parameter combinations for moderate-length ($10^2 < Z < 10^4$) cylinders buckling in the dominant mode. By concentrating on the buckling equations (within the assumptions of the theory) we have avoided some of the conclusions made in References 4 or 6, i.e., "that axisymmetric buckling occurs for 'short' cylinders" or "that axisymmetric buckling cannot occur for ring-stiffened shells with inside stiffeners".

This a priori knowledge of buckling modes is extremely useful in the computationally efficient integration schemes for the related problem of the dynamic stability of stiffened shells.